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THE ACTION OF THE AUTOMORPHISM GROUP OF F_2 UPON THE A_6- AND PSL(2, 7)-DEFINING SUBGROUPS OF F_2

BY

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ABSTRACT. In this paper is described a graphical technique for determining the action of the automorphism group Φ_2, of the free group F_2 of rank 2 upon those normal subgroups of F_2 with quotient groups isomorphic to G, where G is a group represented faithfully as a permutation group. The procedure is applied with $G = \mathrm{PSL}(2,7)$ and A_6 (the case $G = A_5$ having been treated in an earlier paper) with the following results:

Theorem 1. *Φ_2 acts upon the* 57 *subgroups of F_2 with quotient isomorphic to* PSL(2,7) *with orbits of lengths* 7, 16, 16, *and* 18. *The action of Φ_2 is that of A_{16} in one orbit of length* 16, *and of symmetric groups of appropriate degree in the other three orbits.*

Theorem 2. *Φ_2 acts upon the* 53 *subgroups of F_2 with quotients isomorphic to A_6 with orbits of lengths* 10, 12, 15, *and* 16. *The action is that of full symmetric groups of appropriate degree in all orbits.*

1. **Introduction.** In [1], the author calculated the action of Φ_2, the automorphism group of F_2, (the free group of rank 2) upon the 19 A_5-defining subgroups of F_2 (i.e., the normal subgroups N of F_2 with $F_2/N \cong A_5$), where A_5 is the alternating group of degree 5. The action of Φ_2 upon these subgroups was found to have orbits of lengths 9 and 10, and Φ_2 acts as the full symmetric group (of appropriate degree) upon each orbit.

In this paper, the technique of [1] is applied, with some modifications, to the PSL(2,7)- and A_6-defining subgroups of F_2 (of which there are respectively 57 and 53) to arrive at the following results:

Theorem 1. *The action of Φ_2 upon the* PSL(2,7)*-defining subgroups of F_2 has orbits of lengths* 7, 16, 16, *and* 18. *Φ_2 acts as a full alternating group upon one orbit of length* 16 *and as a full symmetric group upon the other three.*

Theorem 2. *The action of Φ_2 upon the A_6-defining subgroups of F_2 has orbits of lengths* 10, 12, 15, *and* 16. *Φ_2 acts as a full symmetric group upon each orbit.*

2. **Calculation technique.** In this section, we describe without proofs the calculation technique developed in [1].

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Let G be a transitive permutation group of degree j such that G can be generated by the ordered pair of permutations (r, t). Assume that G acts upon the set $\{1, \cdots, j\}$. Let F_2 have free generators a and b. Define $H_i = \{$all words $W(a, b) \in F_2 |$ the permutation $W(r, t)$ fixes $i\}$, $i = 1, \cdots, j$. Then $F_2/\bigcap_{i=1}^{j} H_i \cong G$; that is, $N(r, t) = \bigcap_{i=1}^{j} H_i$ is a G-defining subgroup of F_2.

Proposition 1. *Let $\phi \in \Phi_2$. Write $W_a(a, b) = \phi(a)$, $W_b(a, b) = \phi(b)$. Then $W_a(r, t)$ and $W_b(r, t)$ generate G and $N(r, t) = \phi(N(W_a(r, t), W_b(r, t)))$.*

(See [1] for proof.)

Draw a graph $\Gamma(r, t)$ as follows: Pick two different colors, and call one "color a" and the other "color b". Label j points with the numbers $1, \cdots, j$. Connect vertex i to vertex k by an edge of color a oriented from vertex i to vertex k if and only if $r(i) = k$. Form t-connections with color b similarly. Call $\Gamma(r, t)$ a (2-color) G-defining graph. Two such graphs are said to be isomorphic if there is a 1-1 correspondence between the sets of vertices of the two graphs and between their sets of edges which preserves color and the relations "is the initial vertex of" and "is the terminal vertex of".

Proposition 2. *Let $\Gamma(r, t)$ and $\Gamma(r_1, t_1)$ be isomorphic G-defining graphs. Then $N(r, t) = N(r_1, t_1)$.*

(See [1].)

Let Φ_2 act upon the 2-color G-defining graphs in this fashion: if $\phi(a) = W_a(a, b)$ and $\phi(b) = W_b(a, b)$, then $\phi(\Gamma(r, t)) = \Gamma(W_a(r, t), W_b(r, t))$. From Propositions 1 and 2, if $\phi(\Gamma(r, t)) = \Gamma(r_1, t_1)$, then $N(r, t) = \phi(N(r_1, t_1))$.

The converse to Proposition 2 is not in general true; however, the converse fails only if the action of Φ_2 upon the set of G-defining graphs is imprimitive.

Proposition 1 is used with respect to a generating set of Φ_2. Φ_2 has generators $U(a \to ab, b \to b)$, $P(a \to b, b \to a)$, and $\sigma(a \to a^{-1}, b \to b)$. The objects we are permuting in this paper are normal subgroups of F_2, so σ is superfluous, since it differs by an inner automorphism from a word in U and P: namely, $\sigma = PUPU^{-1}PU \cdot PU\sigma U\sigma P$.

3. **General computing plan.** It is desired to compute the action of Φ_2 upon the G-defining subgroups of F_2.

First, pick a generating pair (r, t) for G. Then systematically apply words in P and U to $\Gamma(r, t)$ until it is not possible to produce G-defining graphs which are not isomorphic to previously produced graphs. Such a collection of G-defining graphs is an orbit of the action of Φ_2 upon the full set of G-defining graphs.

If possible, find a generating pair (r_1, t_1) for G such that $\Gamma(r_1, t_1)$ is not isomorphic to any graph previously produced and generate its orbit with respect to the action of Φ_2 as described in the previous paragraph.

Continue this procedure until all orbits of G-defining graphs, together with the effect of P and U upon them, have been found. If m G-defining graphs have been found, consider P and U as permutations of degree m and draw $\Gamma(U, P)$. It is a graph with as many connected components as there are orbits in the action of Φ_2 upon the G-defining graphs. What can be done with these graphs will be described in the following sections.

4. **Application to PSL(2,7).** Here we sketch the application of §3 with $G =$ PSL(2,7), omitting details of calculation.

PSL(2,7) is a simple group of order 168 which can be represented as the group of linear fractional transformations $x \longrightarrow (ax + b)/(cx + d)$ of GF(7) with a, b, c, d in GF(7) and $ad - bc = 1$. As such, PSL(2,7) is a transitive permutation group on the set GF(7) $\cup \{\infty\}$.

Any pair of noncommuting elements of order 7 in PSL(2,7) is a generating set. (Order in PSL(2,7) is determined by the value of $(a + d)^2$; if $g: x \longrightarrow (ax + b)/(cx + d)$ $\in$ PSL(2,7), then $o(g) = 2$ iff $a + d = 0$, $o(g) = 3$ iff $(a + d)^2 = 1$, $o(g) = 4$ iff $(a + d)^2 = 2$, and $o(g) = 7$ iff $(a + d)^2 = 4$.) Thus, $r: x \longrightarrow x + 1$ and $t: x \longrightarrow x/(x + 1)$ generate PSL(2,7). In permutation form, these elements are $(0\ 1\ 2\ 3\ 4\ 5\ 6)$ and $(1\ 4\ 5\ 2\ 3\ 6\ \infty)$, respectively. The length of the orbit of $\Gamma(r, t)$ is 16.

PSL(2,7) is also generated by any choice of an element of order 7 for r and an element of order 2 for t. Only one graph of this type, namely $\Gamma((1\ \infty\ 6\ 3\ 2\ 5\ 4),$ $(0\ 3)(1\ \infty)(2\ 6)(4\ 5))$, appeared in the first orbit. The pair $r_1: x \longrightarrow x + 1$, $t_1: x \longrightarrow (x - 1)/(2x - 1)$, correspond to a graph, $\Gamma((0\ 1\ 2\ 3\ 4\ 5\ 6), (0\ 1)(2\ 5)(3\ 6)(4\ \infty))$, not isomorphic to that one. The orbit of $\Gamma(r_1, t_1)$ has length 18.

The third orbit, of length 16, was generated starting with another pair $x \longrightarrow x + 1$, $x \longrightarrow (x + 4)/(3x - 1)$ of the "7 and 2" type.

Two elements of order 4 generate PSL(2,7) if their squares do not commute. A search for pairs of this type for which the corresponding graphs had not been previously produced resulted in a generating pair for the fourth orbit, which is of length 7.

The effect of P and U is recorded in Table 1. The numbers were assigned to the graphs in order of their production.

The action of Φ_2 on each orbit is primitive. (§6 contains a discussion of primitivity.) Further, the graphs for the two orbits of length 16 are not isomorphic. Thus, if U is replaced by U^{-1} in the heading of Table 1, the table describes the action Φ_2 upon the PSL(2,7)-defining subgroups of F_2, since there are exactly 57 PSL(2,7)-defining subgroups in F_2 (P. Hall [2]).

A theorem of Jordan (as quoted in Wielandt [3, p. 40]) states that if a transitive, primitive permutation group G of degree n contains a p-cycle, where p is a prime $\leq n - 3$, then G contains A_n. In each of the four transitive groups generated by U and P indicated in Table 1, the hypotheses are satisfied (the last part by U^{12}, U^{12},

U^{12}, and U^4, respectively). Hence, U and P generate A_{16} in orbit 1 and S_{18}, S_{16}, and S_7 in the remaining orbits.

5. **Application to A_6.** Here we sketch the application of §3 with $G = A_6$, omitting details of calculation.

The theorem of Jordan quoted in the previous section was used to find starting pairs for the orbits. The starting pairs used were ((1 2 3 4 5), (1 2 6)), ((1 2 3 4 5), (1 3 6)), ((1 2)(3 4 5 6), (1 3)(5 6)), and ((1 2 3 4 5), (1 6)(2 3)) for orbits of lengths 24, 30, 32, and 20, respectively. The action of Φ_2 upon each of these orbits is imprimitive and reduces to action upon 12, 15, 16, and 10 blocks, all of length 2. The reduced action is primitive in all cases, and the theorem of Jordan shows that the action is that of full symmetric groups in all cases. Since $12 + 15 + 16 + 10 = 53$, all A_6-defining subgroups of F_2 have been treated. (See Tables 2–4 for the results of calculations.)

6. **Primitivity.** The imprimitivity of the action of Φ_2 upon the A_6-defining graphs is easy to recognize once the graphs $\Gamma(U, P)$ are drawn, since these graphs have obvious nontrivial self-isomorphisms, which are indicated in Table 3. (However, the absence of nontrivial self-isomorphisms does not imply primitivity. For example, if $r = (1\ 2)(3\ 4\ 5\ 6)$, $t = (2\ 3\ 5)$, the group generated by r and t—a metacyclic group of order 36—is imprimitive on $\{1, 2, 3, 4, 5, 6\}$, but $\Gamma(r, t)$ has no nontrivial self-isomorphism.)

In the first orbit of the action of Φ_2 upon the PSL(2,7)-defining graphs, the cyclic subgroup generated by U acts intransitively with orbits of length 2, 3, 4, and 7. Since a block of Φ_2 is a block of this subgroup, it is helpful to find the nontrivial blocks of U. They are the orbits, all possible unions of orbits, and the sets $\{8, 10\}$, $\{9, 11\}$, $\{8, 10, 15\}$, $\{9, 11, 16\}$, $\{8, 10, 16\}$, $\{9, 11, 15\}$. Since the size of a block of a transitive group must divide the degree of the group, the only possibilities for nontrivial blocks of Φ_2 are the sets $\{8, 10\}$, $\{9, 11\}$, $\{15, 16\}$, and $\{8, 9, 10, 11\}$. A complete block system cannot be formed from among these, so the action is primitive.

The primitivity of the action for the remaining three orbits for PSL(2,7) and the four reduced orbits for A_6 can be shown in a similar fashion.

7. **Conclusion.** A question suggested by the above results is whether there are groups G for which the action of Φ_2 on the G-defining subgroups of F_2 is not described by full alternating or symmetric groups. The answer is affirmative; F_2 contains 6 C_4-defining subgroups (C_4 = cyclic group of order 4) and Φ_2 acts transitively upon this set. However, the permutation group of degree 6 which describes this action faithfully has S_3 as a quotient group and therefore cannot be A_6 or S_6.

Table 1. The action of Φ_2 on the PSL(2,7)-defining graphs

orbit	U	P
1	(1 2 3 4 5 6 7)(8 9 10 11)(12 13 14)(15 16)	(2 8)(3 11)(5 12)(6 16)(7 14)(13 15)
2	(2 3 4 5 6 7 8)(9 10 11 12)(13 14 15)(16 17 18)	(1 2)(4 9)(5 13)(6 16)(7 12)(10 17)(11 15)(14 18)
3	(1 2)(3 4 5 6 7 8 9)(10 11 12 13)(14 15 16)	(1 3)(2 10)(4 11)(5 15)(8 16)
4	(1 2 3 4)(5 6 7)	(2 5)(3 7)

Table 2. The action of Φ_2 on the A_6-defining graphs

orbit	U	P
1	(1 2 3)(4 5 6 7 8)(9 10 11 12) (13 14 15 16 17)(18 19 20 21)(22 23 24)	(1 4)(2 9)(3 13)(5 10)(6 18)(7 22) (11 21)(12 17)(15 23)(16 20)(19 24)
2	(1 2 3)(4 5 6 7 8)(9 10 11)(12 13 14 15) (16 17 18)(19 20 21 22)(23 24 25 26 27) (28 29 30)	(1 4)(2 9)(3 12)(5 16)(6 19)(7 23) (8 20)(10 22)(11 25)(13 27)(14 24) (15 17)(18 28)(21 30)(26 29)
3	(1 2)(3 4 5 6)(7 8 9 10 11)(12 13 14 15 16) (17 18 19 20 21)(22 23 24 25 26) (27 28 29 30)(31 32)	(1 3)(2 7)(4 8)(5 12)(6 17)(9 22) (10 25)(11 27)(13 20)(14 23)(16 21) (18 31)(19 29)(26 30)
4	(1 2)(3 4 5 6 7)(8 9 10 11 12)(13 14 15 16) (17 18 19 20)	(1 3)(2 8)(4 9)(5 13)(6 16)(10 17) (11 20)(14 18)

Table 3. Blocks of the action of Φ_2 upon A_6-defining graphs

orbit → block ↓	1	2	3	4
A	1, 23	1, 11	1, 32	1, 2
B	2, 24	2, 9	2, 31	3, 8
C	3, 22	3, 10	3, 28	4, 9
D	4, 15	4, 25	4, 29	5, 10
E	5, 16	5, 26	5, 30	6, 11
F	6, 17	6, 27	6, 27	7, 12
G	7, 13	7, 23	7, 18	13, 17
H	8, 14	8, 24	8, 19	14, 18
I	9, 19	12, 22	9, 20	15, 19
J	10, 20	13, 19	10, 21	16, 20
K	11, 21	14, 20	11, 17	
L	12, 18	15, 21	12, 26	
M		16, 29	13, 22	
N		17, 30	14, 23	
O		18, 28	15, 24	
P			16, 25	

Table 4. Action of Φ_2 upon A_6-defining subgroups

orbit	U^{-1}	P
1	$(A\ B\ C)(D\ E\ F\ G\ H)(I\ J\ K\ L)$	$(A\ D)(B\ I)(C\ G)(E\ J)(F\ L)$
2	$(A\ B\ C)(D\ E\ F\ G\ H)(I\ J\ K\ L)$ $(M\ N\ O)$	$(A\ D)(C\ I)(E\ M)(F\ J)(H\ K)$ $(L\ N)$
3	$(A\ B)(C\ D\ E\ F)(G\ H\ I\ J\ K)$ $(L\ M\ N\ O\ P)$	$(A\ C)(B\ G)(D\ H)(E\ L)(F\ K)$ $(I\ M)(J\ P)$
4	$(B\ C\ D\ E\ F)(G\ H\ I\ J)$	$(A\ B)(D\ G)(E\ J)$

We then ask: For which G is the action of Φ_2 described by full alternating or symmetric groups? The results for $A_5 = \mathrm{PSL}(2,5)$, PSL (2,7), and A_6 suggest a search through the sequences A_n and PSL(2, p) (p prime) of simple groups, particularly the latter, since the number of PSL(2, p)-defining subgroups of F_2 is given for all p in [2]. Since the number of PSL(2,11)-defining subgroups of F_2 is already 254, hand calculation should give way to the use of an electronic computer.

Also, is orbit size predictable without such extensive calculation? What is the structure of the intransitive groups whose transitive constituents have been calculated?

The above questions provide material for further research.

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